Free fall and the equivalence principle revisited

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Abstract. Free fall is commonly discussed as an example of the equivalence principle, in the context of a homogeneous gravitational field, which is a reasonable approximation for small test masses falling moderate distances. Newton's law of gravity provides a generalisation to larger distances, and also brings in an inhomogeneity in the gravitational field. In addition, Newton's third law of action and reaction causes the Earth to accelerate towards the falling object, bringing in a mass dependence in the time required for an object to reach ground - in spite of the equivalence between inertial and gravitational mass. These aspects are rarely discussed in textbooks when the motion of everyday objects are discussed. Although these effects are extremely small, it may still be important for teachers to make assumptions and approximations explicit, to be aware of small corrections, and also to be prepared to estimate their size. Even if the corrections are not part of regular teaching, some students may reflect on them, and their questions deserve to be taken seriously.

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1. Introduction

Physics teaching - in schools and universities - is full of assumptions and simplifications that vary between contexts, but are often not discussed and made explicit. Going between different sets of tacitly agreed rules can be seen as "games physicists play" This can be confusing for students, who may perceive contradictions unless the relations between them and reason for approximations made in different contexts are discussed. Redish [1] notes that "Knowing what to ignore is an important part of learning to think about science, and it should not be treated as trivial". This paper addresses the negligibly small effect on the motion of the Earth by a falling object, and was prompted by a recent paper by Spallicci and van Putten [2]. The influence of air, which is the most obvious effect influencing the time for an object to reach ground [3], will not be discussed in this paper, nor will other observable effects on the fall, such as the influence of the rotation of the Earth on falling objects [4, 5].

Free fall is commonly introduced in terms of acceleration of gravity **g**, which could be seen as describing a homogeneous gravitational field. The same holds for other textbook discussions about the motion of everyday objects, and even discussions about Newton's law of action and reaction often fail to discuss explicitly the gravitational force from an object acting on the earth.

A homogeneous gravitational field is a very good approximation for small test masses falling moderate distances, and is sufficient for discussions of the challenge to the everyday experience of light objects falling slower that heavier ones. Newton's law of gravity provides a generalisation to larger distances, where g is replaced by GM/r^2. The law of gravity also brings in an inhomogeneity in the gravitational field, causing e.g. tidal forces on the Earth, but also on astronauts on the space station. The textbook connection between the descriptions is typically limited to applying Newton's law of gravity to calculate a value for $g = GM/R^2$ at the surface of the Earth with mass M and radius R.

The effect of the Earth's rotation on the shape of the earth was discussed by Newton in his Principia [6], and students usually know that the radius of the Earth is slightly larger if measured at the equator than at the poles, and that the rotation leads to small deviations in the values for g.

What seems not to be discussed at all in most textbooks is the observation that Newton's third law of action and reaction causes the Earth to accelerate towards the falling object, bringing in a mass dependence in the time required for an object to reach ground (figure 1). The effect is, of course, very small, and can safely be neglected from an experimental point of view. Nevertheless, it is a fundamental insight, that students should be given the opportunity to share, and which brings coherence between areas of physics that are traditionally taught separately. Lehavi and Galili [7] have interviewed a number of teachers and students and found that the threshold to this insight was quite high: "The scheme of knowledge (a phenomenological primitive) of a motionless Earth amazingly prevailed over the requirements of the Third Newton's law."

In their recent paper, Spallicci and van Putten [2] review how free fall is discussed both in physics education literature and high-impact-factor journals. They go as far as characterising teaching that fails to mention these approximations as "sloppy teaching" and "brainwashing". They also argue that students should learn e.g. that "An observer comoving with the center of mass of the system (stone plus Earth) would observe 1 kg mass falling faster than a 2 kg mass." (sic!) In this paper we discuss the free fall in a Newtonian framework and arrive at a different conclusion.

2. Falling objects and the acceleration of the Earth

We consider now the acceleration of the Earth (mass $M$and radius R) caused by the interaction between a falling body with mass m at a distance h over the surface of the

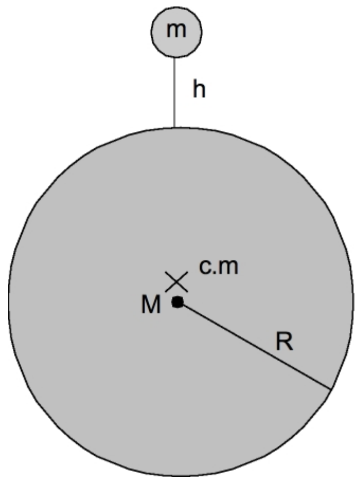

Figure 1. The Earth with mass M and radius R and a falling body with mass m at a distance h over the surface of the Earth. The force between them, $F = GmM/(R+h)^2$, causes the centre of mass of the Earth to accelerate, whereas the joint centre of mass does not. (Here, of course, we omit the interactions with sun, moon and other bodies.)

Earth (figure 1). From Newton's law of gravity we know the size of the force between the body and the Earth:

$$F = G\frac{Mm}{(R+h)^2} \tag{1}$$

Using Newton's second law, we find the usual expression for the acceleration of the falling body: $a_m = -GM/(R+h)^2$, where the negative sign indicates the direction from mass m towards the centre of the Earth. Similarly, the acceleration of the Earth becomes $a_E = Gm/(R+h)^2$. Combining these expressions we find the relative acceleration

$$a_{rel} = a_m - a_E = -G\frac{M+m}{(R+h)^2} = -G\frac{M(1+m/M)}{(R+h)^2}. \tag{2}$$

The acceleration of the Earth thus brings an extra factor $(1 + m/M)$. This has the remarkable consequence that the time to reach the ground depends on the mass of the object. *Relative to the Earth* a heavier object would accelerate faster and thus hit the ground after a shorter time. Of course, the large mass of the Earth ($M \approx 6 \times 10^{24} kg$) makes the ratio m/M far too small to have any practical consequences.

During the fall, the acceleration increases as the object approaches the Earth. This increase will not be discussed below. It does not influence the arguments, since for any distance between the object and the centre of the Earth, the *relative acceleration* carries an extra factor $(1 + m/M)$ due to the acceleration of the Earth.

However, in this discussion about the time for different masses to reach the ground, the objects were considered to be dropped separately. What happens if both objects are dropped simultaneously, next to each other?

2.1. Two objects falling together

When evaluating small corrections, direct comparison between the corrections is often a powerful method. The legendary Galileo experiment involves dropping two bodies with different masses, m_1 and m_2 simultaneously from a tower. The interaction between the Earth and each of the masses is given by Eq. (1). Neglecting the gravitational interaction between the two bodies, we see that the accelerations of m_1 and m_2 are unchanged. (Also neglected are the inhomogeneities of the gravitational field over the extension of these bodies.) The force, F_E, on the Earth is given by the sum of the forces, F_1 and F_2, from the two masses, and we find

$$F_E = F_1 + F_2 = G\frac{Mm_1}{(R+h)^2} + G\frac{Mm_2}{(R+h)^2} = G\frac{M(m_1+m_2)}{(R+h)^2} \tag{3}$$

The acceleration of the Earth thus becomes larger than for either of the masses falling separately:

$$a_E = G\frac{(m_1+m_2)}{(R+h)^2} \tag{4}$$

Adding a second falling body leads to a larger acceleration relative to Earth for both of the falling objects than for either falling individually, but the relative acceleration is the same for both bodies.

$$a_{rel} = -G\frac{M}{(R+h)^2} - G\frac{m_1+m_2}{(R+h)^2} = -G\frac{M(1+(m_1+m_2)/M)}{(R+h)^2} \tag{5}$$

Thus, a direct comparison of the simultaneous fall of two objects can not reveal that the accelerations relative to the Earth would be different if the objects were dropped individually.

3. Motion in the centre-of-mass system

In this work we let the imaginary observer remain in the joint centre-of-mass system of the earth and the mass that is falling. We choose an observation point which is on the surface of Piazza dei Miracoli, before the mass m is raised until it reaches the top of the tower of Pisa, at a distance h above *the surface of the Earth*. The force responsible for lifting the mass also pushes the Earth in the opposite direction, according to Newton's third law. Thus, when the mass m has moved a distance S in the centre-of-mass system, the Earth with mass M has moved away from the joint centre of mass by the much smaller distance $\Delta r_M = (m/M)S$, leaving the imaginary observer at this height over the ground at Piazza dei Miracoli. (The small mass m is then a total of $(1 + m/M)S$ over the surface of the Earth.) The motion of the Earth also brings the top of the tower closer to the joint centre of mass, and after the mass has moved by S, the remaining distance to the top of the tower will be $h - S - \Delta r_M = h - S(1 + m/M)$. This analysis shows that the mass m has reached the top of the tower after moving a mass-dependent distance $S = hM/(M + m)$ in the centre of mass system, while the Earth has moved away the much shorter distance, $\Delta r_M = hm/(M + m)$.

Since moving a heavier body leads to a larger shift of the Earth relative to the centre of mass, the heavier body needs to fall a shorter distance before hitting the earth. With a distance $(R+h)$ from the center of the earth, either mass, m_1 or m_2, will accelerate by $GM/(R+h)^2$ towards the centre of mass (and the contact point), but if they are dropped separately, the heavier object moves a shorter distance than the lighter object before hitting the ground at Piazza dei Miracoli.

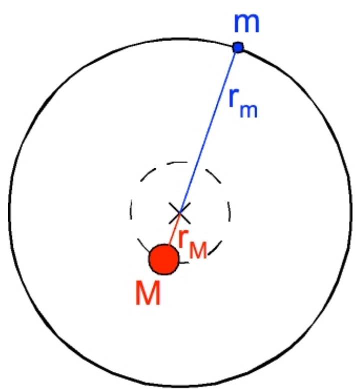

Figure 2. Illustrations of "planetary orbits" of an object with mass m, around a heavier body with mass M. Their distances to the centre of mass are related as $r_M = r_m(m/M)$.

Corrections to a stationary Earth are typically discussed in connection with the lunar motion around the Earth, (see e.g. [8]). Figure 2 shows a lighter object moving in the field from heavier body. The two masses will always be on opposite sides of the centre of mass, at distances related by

$$mr_m = Mr_M. \tag{6}$$

For the case of a body falling from a tower with elevation h above the surface of the earth with radius R, we also find

$$r_m + r_M = R + h \tag{7}$$

before the fall.

Spallicci and van Putten [2] choose to rewrite the gravitational field from the heavier body in terms of coordinate relating to the joint centre-of-mass frame, which gives

$$a_m = -\frac{GM}{(r_m + r_M)^2} = -\frac{GM}{r_m^2\left(1+\frac{m}{M}\right)^2} \approx -\frac{GM\left(1-2\frac{m}{M}\right)}{r_m^2}\ . \tag{8}$$

The next terms in the series expansion can be neglected in view of the small value of m/M. However at this stage, Spallicci and van Putten [2] (p 5) continue by comparing the falls of different masses for the same value of r_m, which leads them to the conclusion that heavier objects fall slower in the centre-of-mass system. By leaving r_m unchanged, their comparison involves drops from mass-dependent heights of the tower, given by $h(m) = h_0(1 + m/(M + m))$, which increases with larger values for m. For an observer in the joint centre of mass system, Spallicci and van Putten thus consider the case of larger masses starting at larger distances from the Earth centre of mass, leading them to a conclusion which differs from the common expression used above, i.e. $a_m = -GM/(R+h)^2$ at the beginning of the fall, whereas their expression for the acceleration as seen from the Earth centre of mass agrees with the expression in (2).

4. Should the equivalence principle be taught in school?

The equivalence between inertial and gravitational mass has many consequences that may be surprising. Our work has shown that teachers and students alike are often fascinated and intrigued by seeing lighter and heavier objects moving side by side. In connection with a follow-up after and amusement park physics day, a teacher told about parents complaining about the 10-year old pupils starting to climb bookshelves to watch objects fall together - but the parental complaints were not about the climbing, but about not knowing how to explain what happened. Other examples when mass does not influence motion are pendulum motion [9], amusement park chain flyer rides [10, 11], liquids in accelerated motion [9, 12, 13] and even objects sliding or rolling down an inclined plane [3, 14]. These examples challenge the very common expectations that heavier objects will, e.g., fall, roll, swing or slide faster.

The fascination in connection with these investigations can be a way to invite teachers (and their students) to go beyond inserting numbers in formulæ, and to take a closer look into the relevant equations and get a glimpse of the more general principle of the equivalence between gravitational and inertial mass.

Spallicci and van Putten [2] argue in their paper that the discussions of free fall and the equivalence principle should always be accompanied by discussions about effect of the motion of the Earth. However, in our work we have found that children can be fascinated by consequences of the equivalence between inertial and gravitational mass long before they are ready for the mathematical descriptions.

Physics teachers in higher education are likely to take the fact that all objects fall equally fast to the ground (or would fall equally fast in vacuum) for granted, but may not have to have reflected on the effect of Newton's third law on the acceleration of objects relative to the Earth. Having taught a number of large cohorts of bright and curious engineering physics students at Chalmers university of technology, I have never heard anyone bring up the question, and confess to not having thought about it until very recently.

Estimating the sizes of small corrections is another game that physicists play, and students may be exhilarated by being introduced to this game. Some students may start this type of game by themselves, and risk encountering ridicule by teachers or peers for considering unrealistic cases. Research about gifted children (see e.g. [15]) shows that they are often bored and even vulnerable in school. This paper aims to prepare teachers to deal with unusual questions from gifted students with special interest, and also provide examples that may feed curiosity.

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I would like to thank A Spallicci for bringing the work in reference [2] to my attention, giving me reason to reconsider the case of objects falling, and also for interesting discussions.

My own fascination with small corrections goes back go post-doc days at University of Washington and University of Oxford, where I worked on calculations of weak-interaction induced atomic parity non-conserving effects [16, 17]. I would also like thank the Institute for Nuclear Theory at University of Washington for the opportunity to return for several visits, and learn about the ongoing search for small deviations from Newton's law of gravity for small distances in the Eöt-Wash project [18], sparking an interest in the Equivalence Principle.

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